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Accurate Monte Carlo Tests of the Stochastic Ginzburg–Landau Model with Multiplicative Colored Noise

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A accurate and fast Monte Carlo algorithm is proposed for solving the Ginzburg–Landau equation with multiplicative colored noise. The stable cases of solution for choosing time steps and trajectory numbers are discussed.

KEY WORDS: Stochastic differential equation; multiplicative colored noise; numerical tests; Ginzburg–Landau model.

Traditionally, the Fokker–Planck equation of high dimensionality is difficult to solve numerically, while the computer simulation (Monte Carlo method) of the equivalent nonlinear stochastic differential equation offers a powerful technique for obtaining information in the same context. There have been many substantial contributions to the subject, exemplified by the work of Fox.[1–3] A typical example is the simplest Ginzburg–Landau model[4] with finiter bandwidth noise source defined as

$$\dot{x} = f(x) + g(x)\, y(t) \tag{1}$$

$$\dot{y} = -\frac{1}{\tau_c}\, y + \frac{(2D)^{1/2}}{\tau_c}\, \xi(t) \tag{2}$$

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\, \xi(t') \rangle = \delta(t - t') \tag{3}$$

$\xi(t)$ is Gaussian white noise, τ_c is the colored noise correlation time, and D is a measure of the strength of the noise. Let $f(x) = x - x^3$, $g(x) = x$.

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The stable cases of numerical solution of these equations have not been studied with the same detail as in the cases of white-noise or additive colored-noise processes.[5,6]

The algorithm presented here differs from previous approaches[4–7] in four ways:

(i) The solution of the stiff equation (2) is obtained in closed from with no approximation.[2]

(ii) Taylor expansion of $g(x)$ only to the first order locally is needed to satisfy the optimal result when the stochastic order is equal to that of the deterministic one.[8]

(iii) A semi-implicit technique for $f(x)$ can give the greatest accuracy.

(iv) The output results are average values of calculated points in the platform of the time steps.

We begin the derivation by formally integrating Eqs. (1) and (2),

$$y(t+h) = \exp(-h/\tau_c)\, y(t) + (2D)^{1/2}/\tau_c w_0(t, h) \tag{4}$$

$$x(t+h) = x(t) + \int_t^{t+h} f(x(t'))\, dt' + \int_t^{t+h} g(x(t'))\, y(t')\, dt \tag{5}$$

where

$$w_0 = \int_t^{t+h} \exp\left[\frac{s-t-h}{\tau_c}\right] \xi(s)\, ds \tag{6}$$

Now we expand only $g(x(t'))$ as follows:

$$g(x(t')) = g(x(t)) + \frac{\partial g}{\partial x}\,[x(t') - x(t)] \tag{7}$$

At the lowest order $o(h)$, we have from (5)

$$x(t') - x(t) = f(x(t))(t'-t) + g(x(t)) \int_t^{t'} y(s)\, ds \tag{8}$$

Substituting (7) back into (5), we obtain

$$x(t+h) = x(t) + f(x(t))h + g(x(t))\, Z_1(t, h) + g(x(t)) \frac{\partial g}{\partial x} Z_2(t, h) \tag{9}$$

Note that in Eq. (8), we take only terms of order h for $\tau_c \to 0$, which corresponds to taking the limit of white noise, and where

$$Z_1(t, h) = \int_t^{t+h} y(t')\, dt' = \tau_c(1 - e^{-h/\tau_c})\, y(t) + \frac{(2D)^{1/2}}{\tau_c} w_1 \tag{10}$$

Here

$$w_1 = \int_t^{t+h} dt' \int_t^{t'} \exp\left[\frac{s-t'}{\tau_c}\right] \xi(s)\, ds \tag{11}$$

and

$$\begin{aligned} Z_2(t,h) &= \int_t^{t+h} dt'\, y(t') \int_t^{t'} y(s)\, ds \\ &= \frac{\tau_c^2}{2}(1-e^{-h/\tau_c})^2\, y^2(t) + (2D)^{1/2}(1-e^{-h/\tau_c})\, y(t) w_1 \\ &\quad + \frac{2D}{\tau_c^2}\int_t^{t+h} dt'\, w_0(t')\, w_1(t') \end{aligned} \tag{12}$$

In general,

$$\int_t^{t+h} dt'\, w_0(t')\, w_1(t') = \tfrac{1}{2}[w_1^2 - p\langle w_1^2\rangle] \tag{13}$$

where $p=0$ (1) results in the Stratonovich (Itô) form.

If R_1, R_2 are two uncorrelated Gaussian variables with average zero and standard deviation one, they are generated by the Box–Müller formula,

$$w_0 = \langle w_0^2\rangle^{1/2} R_1 \tag{14}$$

$$w_1 = \frac{\langle w_0 w_1\rangle}{\langle w_0^2\rangle^{1/2}} R_1 + \left[\langle w_1^2\rangle - \frac{\langle w_0 w_1\rangle^2}{\langle w_0^2\rangle}\right]^{1/2} R_2 \tag{15}$$

The variance and the cross correlations of w_0 and w_1 are readily determined from their definitions (see ref. 6).

The final algorithm reads:

(1) Predictor:

$$\begin{aligned} y(t+h) &= \exp(-h/\tau_c)\, y(t) + (2D)^{1/2}/\tau_c w_0 \\ \tilde{x}(t+h) &= x(t) + hf(x(t)) \\ &\quad + g(x(t))\{\tau_c[1-\exp(-h/\tau_c)]\, y(t) + (2D)^{1/2}/\tau_c w_1\} \\ &\quad + [g(x(t))/2](\partial g/\partial x)\{\tau_c^2[1-\exp(-h/\tau_c)]^2\, y^2(t) \\ &\quad + 2(2D)^{1/2}[1-\exp(-h/\tau_c)]\, y(t) w_1 \\ &\quad + (2D/\tau_c^2)(w_1^2 - p\langle w_1^2\rangle)\} \end{aligned} \tag{16}$$

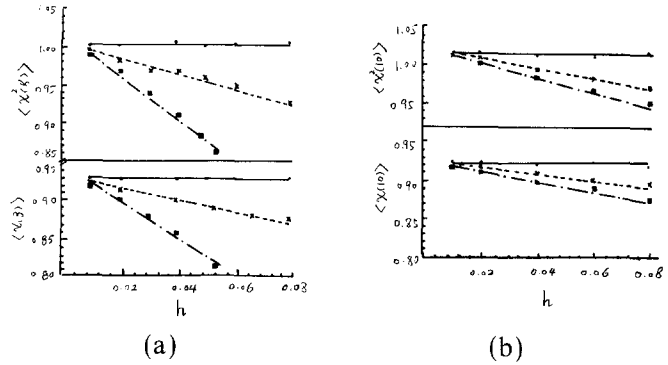
(2) Corrector:

$$x(t+h) = x(t) + (h/2)[f(x(t)) + f(\tilde{x}(t+h))] + g(x(t))\{\tau_c[1-\exp(-h/\tau_c)]\, y(t) + (2D)^{1/2}/\tau_c w_1\} + [g(x(t))/2](\partial g/\partial x)\{\tau_c^2[1-\exp(-h/\tau_c)]^2\, y^2(t) + 2(2D)^{1/2}[1-\exp(-h/\tau_c)]\, y((t)w_1 + (2D/\tau_c^2)(w_1^2 - p\langle w_1^2\rangle)\} \quad (17)$$

Note that w_0 and w_1 used in (16) are the same as those of (17). All of the cited algorithms explicitly invoke the Stratonovich choice and not the Itô choice, which requires that p satisfy $p=0$.

The aim of our simulations is the determination of the first and second moments at the stationary state, which are computed and averaged over 20,000 distinct trajectories, starting from $\{x(0)=0.5,\ y(0)$ according to the appropriate Gaussian distribution$\}$. The numerical results of the three methods are shown in Figs. 1a and 1b for two groups of parameters: $D=0.5$, $\tau_c=0.3$, and $D=1.5$, $\tau_c=2$.

The results show that the sampling error in the moments is strongly dependent on the algorithm chosen. The semi-implicit algorithm in the present study generally has the lowest error and allows a much larger time step. By increasing the step size to 0.06 and 0.09, we find that the algorithm of ref. 4 leads to numerical overflow in the above two groups of parameters, respectively. It is evident that our algorithm has a range of convergence far larger than either the algorithm of ref. 4 or the Euler method, and the white-noise limit (i.e., $\tau_c \to 0$, h finite, $h/\tau_c \to \infty$) can be safely taken.



(a) (b)

Fig. 1. The numerical results of the first and second moments vs. the time step. Circles, algorithm of ref. 4; crosses, Euler method; squares, algorihtm of this paper. (a) $D=0.5$, $\tau_c=0.3$, (b) $D=1.5$, $\tau_c=2$.

Table I. The Simulation Results vs. Trajectory Number ($h = 0.05$)

$D = 0.5$, $\tau_c = 0.3$, time = 8			$D = 1.5$, $\tau_c = 2$, time = 10			
N	$\langle x^2 \rangle_{st}$ [a]	$\langle x \rangle_{st}$ [a]	$\langle x^2 \rangle_{st}$ [a]	$\langle x \rangle_{st}$ [a]	$\langle x^2 \rangle_{st}$ [4]	$\langle x \rangle_{st}$ [4]
1000	1.008445	0.9306765	1.004083	0.916596	1.005249	0.915108
5000	1.015417	0.9335762	1.013147	0.920507	0.992142	0.906477
10000	1.006932	0.9300061	1.014279	0.921361	0.989836	0.903631
15000	1.008554	0.9300036	1.013206	0.920712	0.993527	0.905108
20000	1.005929	0.9282101	1.012981	0.921867	0.989958	0.903341

[a] Present paper.

We have checked our result against Fox's second-order algorithm.[3] For a step size of 0.08, the two algorithms give identical results for an average of over 20,000 realizations. This shows that higher expansions in h may not be necessary from a practical point of view.[5]

Treating stochastic differential equations by the Monte Carlo method, one will have two major contributions to the error, namely the time discretization and another one from the finite number of realizations. Table I shows the solutions versus the trajectory number N. It is seen from Table I that the dependence of the simulation results on the number of realizations can surely be more stable. The output results are average values of each calculated point in the platform of the time step, and they are more accurate and reliable than those from the linear extrapolation method.[5]

In conclusion, the algorithm proposed here gives much smaller errors than other methods, especially when calculating the moments. It is probably the most stable in the case of stiff equations; in addition, it is shorter, runs faster, and permits the use of longer time steps.

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